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Commuting mappings of generalized matrix algebras ☆

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ABSTRACT

In this paper we will describe the general form of commuting mappings of a class called generalized matrix algebras and consider the question of when all commuting mappings of such generalized matrix algebras take a certain form which is said to be proper. These work extend the main results of [8] to the case of generalized matrix algebras. A number of applications related to commuting mappings are presented.

1. Introduction

Let us begin with the definition of generalized matrix algebras given by a Morita context. Let $\mathcal{R}$ be a commutative ring with identity. A Morita context consists of two $\mathcal{R}$-algebras A and B, two bimodules ${}_AM_B$ and ${}_BN_A$, and two bimodule homomorphisms called the pairings $\Phi_{MN} : M \underset{B}{\otimes} N \longrightarrow A$ and $\Psi_{NM} : N \underset{A}{\otimes} M \longrightarrow B$ satisfying the following commutative diagrams:

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$$\begin{tikzcd}
M \underset{B}{\otimes} N \underset{A}{\otimes} M \arrow[r, "\Phi_{MN}\otimes I_M"] \arrow[d, "I_M\otimes\Psi_{NM}"] & A \underset{A}{\otimes} M \arrow[d, "\cong"] \\
M \underset{B}{\otimes} B \arrow[r, "\cong"] & M
\end{tikzcd}$$

and

$$\begin{tikzcd}
N \underset{A}{\otimes} M \underset{B}{\otimes} N \arrow[r, "\Psi_{NM}\otimes I_N"] \arrow[d, "I_N\otimes\Phi_{MN}"] & B \underset{B}{\otimes} N \arrow[d, "\cong"] \\
N \underset{A}{\otimes} A \arrow[r, "\cong"] & N .
\end{tikzcd}$$

Let us write this Morita context as $(A, B, M, N, \Phi_{MN}, \Psi_{NM})$. If $(A, B, M, N, \Phi_{MN}, \Psi_{NM})$ is a Morita context, then the set

$$\begin{bmatrix} A & M \\ N & B \end{bmatrix} = \left\{ \begin{bmatrix} a & m \\ n & b \end{bmatrix} \middle| a \in A, m \in M, n \in N, b \in B \right\}$$

form an $\mathcal{R}$-algebra under matrix-like addition and matrix-like multiplication, where at least one of the two bimodules M and N is distinct from zero. Such an $\mathcal{R}$-algebra is called a *generalized matrix algebra* of order 2 and is usually denoted by $\mathcal{G} = \begin{bmatrix} A & M \\ N & B \end{bmatrix}$. This kind of algebra was first introduced by Sands in [18], where the author investigated various radicals of rings occurring in Morita contexts. Obviously, when $M = 0$ or $N = 0$, $\mathcal{G}$ exactly degenerates to the so-called *triangular algebra*. It should be remarked that our current generalized matrix algebras contain those generalized matrix algebras in sense of Brown [6] as special cases (see below Section 2.4).

Let $\mathcal{R}$ be a commutative ring with identity, A be a unital algebra over $\mathcal{R}$ and $Z(A)$ be the center of A. Let us denote the commutator or the lie product of the elements $a, b \in A$ by $[a, b] = ab - ba$. Recall that an $\mathcal{R}$-linear mapping $\Theta : A \longrightarrow A$ is said to be *commuting* if $[\Theta(a), a] = 0$ for all $a \in A$. A commuting mapping Θ of A is called *proper* if it is of the form

$$\Theta(a) = ac + \Omega(a), \quad \forall a \in A,$$

where $c \in Z(A)$ and Ω is an $\mathcal{R}$-linear mapping from A into $Z(A)$. When we investigate a commuting mapping, the principal task is to describe its form. The identity mapping and every mapping which has its range in $Z(A)$ are two classic examples of commuting mappings. Furthermore, the sum and the pointwise product of commuting mappings are also commuting mappings. The most important result on commuting mappings is Posner's theorem which states that the existence of a nonzero commuting derivation on a prime algebra A implies that A is commutative [17]. Brešar [4] showed that both commuting mappings on simple unital algebras and commuting mappings on prime algebras are proper. We encourage the reader to read the well-written survey paper [4], in which the author presented the development of the theory of commuting mappings and their applications in detail. The following topics are discussed in [4]: commuting derivations, commuting additive mappings, commuting traces of multiadditive mappings, various generalizations of the notion of a commuting mapping, and applications of results on commuting mappings to different areas, in particular to Lie theory.

It was Cheung [7,8] who initiated the study of commuting mappings of matrix algebras. He determined the class of triangular algebras for which every commuting linear mapping is proper. Motivated by the aforementioned results of Brešar and Cheung, Benkovič and Eremita in [1] studied commuting traces of bilinear mappings on triangular algebras. The authors gave conditions under which every commuting trace of a triangular algebra is proper. This is applied to the study of Lie isomorphisms and of commutativity preserving mappings. The cases of upper triangular matrix algebras and nest algebras are considered. Yu and Zhang [20] introduced the notion of σ-commuting mappings and described the general form of every σ-commuting mapping on the nest algebra with $\dim\{0\}_+ \neq 1$ or $\dim H_-^{\perp} \neq 1$. The problem to describe the structure of commuting mappings seems to be unapproachable. The Example 1 of [8] constructed by Cheung shows that in general not much can be said. People pay much less attention to linear mappings of generalized matrix algebras, to the best of our knowledge there are fewer articles dealing with linear mappings of generalized matrix algebras except for [14]. Following the well-established approach and the sophisticated computational method [8] by Cheung we shall give some new characterizations on commuting mappings of generalized matrix algebras. This will make new prospects for the future research work of linear mappings of generalized matrix algebras.

This paper is devoted to the study of commuting mappings of generalized matrix algebras and its framework is as follows. The second section provides some basic examples of generalized matrix algebras which we will work with in later. In the third section, we describe the general form of an arbitrary commuting mapping on generalized matrix algebras (Proposition 3.3) and give some sufficient conditions which enable the commuting mappings to be proper (Proposition 3.5 and Theorem 3.6). Moreover, some applications related to commuting mappings are presented in the last section.

2. Examples of generalized matrix algebras

Let us begin this section with typical examples of generalized matrix algebras which we will use or revisit in the sequel. These generalized matrix algebras mainly come from matrix theory and operator theory.

2.1. A natural construction of generalized matrix algebras

Let $\mathcal{R}$ be a commutative ring with identity and A be a unital algebra over $\mathcal{R}$. Suppose that there exists a nontrivial idempotent $e \in A$. One can easily construct the following generalized matrix algebra:

$$\begin{aligned}\mathcal{G} &= \begin{bmatrix} eAe & eA(1-e) \\ (1-e)Ae & (1-e)A(1-e) \end{bmatrix} \\ &= \left\{ \begin{bmatrix} eae & ec(1-e) \\ (1-e)de & (1-e)b(1-e) \end{bmatrix} \middle| a, b, c, d \in A \right\}.\end{aligned}$$

According to the routine computation, we can verify the $\mathcal{R}$-linear mapping

$$\begin{aligned}\xi : A &\longrightarrow \mathcal{G} \\ a &\longmapsto \begin{bmatrix} eae & ea(1-e) \\ (1-e)ae & (1-e)a(1-e) \end{bmatrix}\end{aligned}$$

is an isomorphism from A to $\mathcal{G}$. Indeed, if

$$\begin{bmatrix} eae & ea(1-e) \\ (1-e)ae & (1-e)a(1-e) \end{bmatrix} = \begin{bmatrix} ebe & eb(1-e) \\ (1-e)be & (1-e)b(1-e) \end{bmatrix},$$

Then $eae = ebe$ and $ea(1-e) = eb(1-e)$. This leads to $ea = eb$. Likewise, we also have $(1-e)ae = (1-e)be$ and $(1-e)a(1-e) = (1-e)b(1-e)$. This gives that $(1-e)a = (1-e)b$. Thus $a = b$ and hence ξ is injective. For any $\begin{bmatrix} eae & ec(1-e) \\ (1-e)de & (1-e)b(1-e) \end{bmatrix} \in \mathcal{G}$, there exists $eae + ec(1-e) + (1-e)de + (1-e)b(1-e) \in A$ such that $\xi(eae + ec(1-e) + (1-e)de + (1-e)b(1-e)) =$

$\begin{bmatrix} eae & ec(1-e) \\ (1-e)de & (1-e)b(1-e) \end{bmatrix}$. So ξ is surjective. Therefore ξ is an isomorphism from A to $\mathcal{G}$. This shows that any unital algebra with nontrivial idempotents is isomorphic to a generalized matrix algebra. Furthermore, simulating the proof of [7, Proposition 1.2.6] we can obtain

Proposition 2.1. *A unital algebra A is a generalized matrix algebra if and only if there exists an idempotent $e \in A$ such that $eA(1-e) \neq 0$.*

Proof. " $\Longleftarrow$ " This is due to the above interpretation.

" $\Longrightarrow$ " Let A be a generalized matrix algebra and let us write $1 = \begin{bmatrix} 1_A & 0 \\ 0 & 1_B \end{bmatrix}$. We assert that one of the following two idempotents $e = \begin{bmatrix} 1_A & 0 \\ 0 & 0 \end{bmatrix}$ and $e' = \begin{bmatrix} 0 & 0 \\ 0 & 1_B \end{bmatrix}$ is the desired idempotent. For all $\begin{bmatrix} a & m \\ n & b \end{bmatrix} \in \begin{bmatrix} A & M \\ N & B \end{bmatrix}$, we have

$$\begin{bmatrix} 1_A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & m \\ n & b \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1_B \end{bmatrix} = \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & 0 \\ 0 & 1_B \end{bmatrix} \begin{bmatrix} a & m \\ n & b \end{bmatrix} \begin{bmatrix} 1_A & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}.$$

Therefore $eA(1-e) \neq 0$. $\square$

2.2. Generalized matrix algebras of order n

In a similar way, we can define a generalized matrix ring of any order $n > 2$. Let $\mathcal{R}$ be a commutative ring with identity and $A_i (i = 1, 2, \ldots, n)$ be unital algebras over $\mathcal{R}$. Let ${}_iM_j$ be nonzero unital (A_i, A_j)-bimodules for $1 \leqslant i \leqslant j \leqslant n$ and ${}_iM_i = A_i$. We observe a family of (A_i, A_k)-bilinear homomorphisms

$$\eta^j_{i,k} : {}_iM_j \bigotimes_{A_j} {}_jM_k \longrightarrow {}_iM_k$$

$$\eta^j_{i,j} : {}_iM_j \bigotimes_{A_j} A_j \cong {}_iM_j$$

$$\eta^i_{i,j} : A_i \bigotimes_{A_i} {}_iM_j \cong {}_iM_j$$

and a family of diagrams

$$\begin{array}{ccc} {}_iM_j \otimes_{A_j} {}_jM_k \otimes_{A_k} {}_kM_l & \xrightarrow{I_{i,j} \otimes \eta^k_{j,l}} & {}_iM_j \otimes_{A_j} {}_jM_l \\ \Big\downarrow {\scriptstyle \eta^j_{i,k} \otimes I_{k,l}} & & \Big\downarrow {\scriptstyle \eta^j_{i,l}} \\ {}_iM_k \otimes_{A_k} {}_kM_l & \xrightarrow{\eta^k_{i,l}} & {}_iM_l \end{array} \qquad (\clubsuit)$$

where $I_{i,j}$ and $I_{k,l}$ denote the identity mappings of ${}_iM_j$ and ${}_kM_l$, respectively. Let us consider the following set

$$\mathcal{G}_n(A_i;\,{}_iM_j) = \left\{ \left[\begin{array}{ccccc} {}_1a_1 & {}_1m_2 & \cdots & {}_1m_{n-1} & {}_1m_n \\ {}_2m_1 & {}_2a_2 & \cdots & {}_2m_{n-1} & {}_2m_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ {}_{n-1}m_1 & {}_{n-1}m_2 & \cdots & {}_{n-1}a_{n-1} & {}_{n-1}m_n \\ {}_nm_1 & {}_nm_2 & \cdots & {}_nm_{n-1} & {}_na_n \end{array} \right] \middle| \, {}_ia_i \in A_i,\, {}_im_j \in {}_iM_j \right\}.$$

We can define the matrix-like addition and matrix-like multiplication on $\mathcal{G}_n(A_i;\,{}_iM_j)$ as below

$$({}_im_j) \pm ({}_im'_j) = ({}_im_j \pm {}_im'_j)$$
$$({}_im_j) \cdot ({}_im'_j) = \left(\sum \eta^j_{i,k}({}_im_j \otimes {}_jm'_k)\right).$$

It is clear that this product is associative if and only if the family of diagrams (♣) are commutative. One can check that $\mathcal{G}_n(A_i;\,{}_iM_j)$ is an $\mathcal{R}$-algebra under the matrix-like addition and the matrix-like multiplication. In this case, $\mathcal{G}_n(A_i;\,{}_iM_j)$ is said to be a *generalized matrix algebra* of order n associated with those bimodules ${}_iM_j(1 \leqslant i \leqslant j \leqslant n)$ and is usually written as

$$\mathcal{G}_n(A_i;\,{}_iM_j) = \left[\begin{array}{ccccc} A_1 & {}_1M_2 & \cdots & {}_1M_{n-1} & {}_1M_n \\ {}_2M_1 & A_2 & \cdots & {}_2M_{n-1} & {}_2M_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ {}_{n-1}M_1 & {}_{n-1}M_2 & \cdots & A_{n-1} & {}_{n-1}M_n \\ {}_nM_1 & {}_nM_2 & \cdots & {}_nM_{n-1} & A_n \end{array} \right].$$

Up to isomorphism, arbitrary generalized matrix algebra of order n is a generalized matrix algebra of order 2. Indeed, if $\mathcal{G}_n(A_i;\,{}_iM_j)$ is a generalized matrix algebra of order n $(n \geqslant 2)$, then there exist $\mathcal{R}$-algebras

$$A = \mathcal{G}_{n-1}(A_i;\,{}_iM_j)(1 \leqslant i \leqslant j \leqslant n-1),\, B = A_n$$

or a nonzero (A, B)-bimodule

$$M = \left[\begin{array}{c} {}_1M_n \\ {}_2M_n \\ \vdots \\ {}_{n-1}M_n \end{array} \right] = \left\{ \left[\begin{array}{c} {}_1m_n \\ {}_2m_n \\ \vdots \\ {}_{n-1}m_n \end{array} \right] \middle| \, {}_im_n \in {}_iM_n,\, 1 \leqslant i \leqslant n-1 \right\}$$

and a nonzero (B, A)-bimodule

$$\begin{aligned} N &= \left[{}_nM_1 \quad {}_nM_2 \quad \cdots \quad {}_nM_{n-1} \right] \\ &= \left\{ \left[{}_nm_1 \quad {}_nm_2 \quad \cdots \quad {}_nm_{n-1} \right] \middle| \, {}_nm_j \in {}_nM_j,\, 1 \leqslant j \leqslant n-1 \right\} \end{aligned}$$

such that

$$\mathcal{G}_n(A_i;\,{}_iM_j) \cong \begin{bmatrix} A & M \\ N & B \end{bmatrix}.$$

A special case of generalized matrix algebras of order n is the case of *tensor generalized matrix algebras* of order n: the modules ${}_sM_r$ with $r - s \geqslant 2$ are tensor products ${}_sM_{s+1} \otimes_{A_{s+1}} \cdots \otimes_{A_{r-1}} {}_{r-1}M_r$. The role of the morphisms $\eta^j_{i,k}$ is played by the identity morphisms of ${}_iM_j \otimes_{A_j} {}_jM_k$, and the associativity of the product of $\mathcal{G}_n(A_i;\,{}_iM_j)$ results from the associativity of the tensor products. In view of the isomorphism relation between generalized matrix algebras of order 2 and generalized matrix algebras of order n $(n > 2)$ and technical considerations, only generalized matrix algebras of order 2 are studied in this paper.

2.3. Full matrix algebras

Let A be a unital $\mathcal{R}$-algebra and $M_n(A)$ be the algebra of $n \times n$ matrices with $n \geqslant 2$. Then the *full matrix algebra* $M_n(A)$ can be represented as a generalized matrix algebra of the form

$$M_n(A) = \begin{bmatrix} A & M_{1\times(n-1)}(A) \\ M_{(n-1)\times 1}(A) & M_{n-1}(A) \end{bmatrix}.$$

2.4. Inflated algebras

Let A be a unital $\mathcal{R}$-algebra and V be an $\mathcal{R}$-linear space. Given an $\mathcal{R}$-bilinear form $\gamma : V \otimes_{\mathcal{R}} V \to A$, we define an associative algebra (not necessarily with identity) $B = B(A, V, \gamma)$ as follows: as an $\mathcal{R}$-linear space, B equals to $V \otimes_{\mathcal{R}} V \otimes_{\mathcal{R}} A$. The multiplication is defined as follows

$$(a \otimes b \otimes x) \cdot (c \otimes d \otimes y) := a \otimes d \otimes x\gamma(b, c)y$$

for all $a, b, c, d \in V$ and any $x, y \in A$. This definition makes B become an associative $\mathcal{R}$-algebra and B is called an *inflated algebra* of A along V. The inflated algebras are closely connected with the cellular algebras which are extensively studied in representation theory. We refer the reader to [13] and the references therein for these algebras.

Let us assume that V is a nonzero linear space with a basis $\{v_1, \cdots, v_n\}$. Then the bilinear form γ can be characterized by an $n \times n$ matrix Γ over A, that is, $\Gamma = (\gamma(v_i, v_j))$ for $1 \leqslant i, j \leqslant n$. Now we can define a new multiplication " $\circ$ " on the full matrix algebra $M_n(A)$ by

$$X \circ Y := X\Gamma Y \quad \text{for all} \quad X, Y \in M_n(A).$$

Under the usual matrix addition and the new multiplication " $\circ$ ", $M_n(A)$ becomes a new associative algebra which is a generalized matrix algebra in the sense of Brown [6]. We denote this new algebra by $(M_n(A), \Gamma)$. It should be remarked that our current generalized matrix algebras contain all generalized matrix algebras defined by Brown [6] as special cases. By [13, Lemma 4.1], the inflated algebra $B(A, V, \gamma)$ is isomorphic to $(M_n(A), \Gamma)$ and hence is a generalized matrix algebra in the sense of ours.

2.5. Upper and lower triangular algebras

Let $\mathcal{R}$ be a commutative ring with identity and A be a unital algebra over $\mathcal{R}$. We denote the set of all $p \times q$ matrices over A by $M_{p\times q}(A)$. Let us denote the set of all $n \times n$ upper triangular matrices over A and the set of all $n \times n$ lower triangular matrices over A by $\mathcal{T}_n(A)$ and $\mathcal{T}'_n(A)$, respectively. For $n \geqslant 2$ and each $1 \leqslant k \leqslant n-1$, the *upper triangular matrix algebra* $\mathcal{T}_n(A)$ and *lower triangular matrix algebra* $\mathcal{T}'_n(A)$ can be written as

$$\mathcal{T}_n(A) = \begin{bmatrix} \mathcal{T}_k(A) & M_{k\times(n-k)}(A) \\ & \mathcal{T}_{n-k}(A) \end{bmatrix} \quad \text{and} \quad \mathcal{T}'_n(A) = \begin{bmatrix} \mathcal{T}'_k(A) & \\ M_{(n-k)\times k}(A) & \mathcal{T}'_{n-k}(A) \end{bmatrix},$$

respectively. Let $\mathbf{H}$ be a complex Hilbert space and $\mathcal{B}(\mathbf{H})$ be the algebra of all bounded linear operators on $\mathbf{H}$. A subalgebra $\mathcal{I}$ of $\mathcal{B}(\mathbf{H})$ is defined to be a *triangular operator algebra* if $\mathcal{D} = \mathcal{I} \bigcap \mathcal{I}^*$ is a maximal abelian self-adjoint subalgebra of $\mathcal{B}(\mathbf{H})$. $\mathcal{D}$ is the *diagonal* of $\mathcal{I}$. To check that $\mathcal{I}$ is a triangular algebra with diagonal $\mathcal{D}$, it is sufficient to show that each self-adjoint operator in $\mathcal{I}$ lies in $\mathcal{D}$ and that $\mathcal{D}$ is contained in $\mathcal{I}$. In fact, $\mathcal{I} \bigcap \mathcal{I}^*$ is a self-adjoint algebra containing each self-adjoint operator in $\mathcal{I}$ and generated linearly by these operators (see [12, Remark 2.1.3]). Any finite dimensional triangular operator algebra is isomorphic to a subalgebra of the upper triangular matrices $\mathcal{T}_n(A)$ which contains the diagonal D_n, for some positive integer n.

2.6. Triangular Banach algebras

Let $(A, \|\cdot\|_A)$ and $(B, \|\cdot\|_B)$ be two given Banach algebras. An algebraic (A, B)-bimodule M is called a Banach (A, B)-bimodule if it is a Banach space and the (A, B)-bimodule actions satisfy $\|am\|_M \leqslant \|a\|_A\|m\|_M$ and $\|mb\|_M \leqslant \|m\|_M\|b\|_B$ for all $a \in A, b \in B$ and $m \in M$. Then the triangular algebra $\mathcal{T} = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$ is a Banach algebra with respect to the norm defined by

$$\left\| \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \right\|_{\mathcal{T}} = \|a\|_A + \|m\|_M + \|b\|_B, \forall \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \in \mathcal{T}.$$

In this case, $\mathcal{T} = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$ is called a *triangular Banach algebra*. It is easy to check that each norm $\| \cdot \|$ making $\mathcal{T} = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$ into a Banach algebra is equivalent to $\| \cdot \|_{\mathcal{T}}$, if the natural restrictions of $\| \cdot \|$ to A, B and M are equivalent to the given norms on A, B and for all M, respectively.

2.7. Nest algebras

Let $\mathbf{H}$ be a complex Hilbert space and $\mathcal{B}(\mathbf{H})$ be the algebra of all bounded linear operators on $\mathbf{H}$. Let I be a index set. A *nest* is a set $\mathcal{N}$ of closed subspaces of $\mathbf{H}$ satisfying the following conditions:

(1) $0, \mathbf{H} \in \mathcal{N}$;
(2) If $N_1, N_2 \in \mathcal{N}$, then either $N_1 \subseteq N_2$ or $N_2 \subseteq N_1$;
(3) If $\{N_i\}_{i\in I} \subseteq \mathcal{N}$, then $\bigcap_{i\in I} N_i \in \mathcal{N}$;
(4) If $\{N_i\}_{i\in I} \subseteq \mathcal{N}$, then the norm closure of the linear span of $\bigcup_{i\in I} N_i$ also lies in $\mathcal{N}$.

If $\mathcal{N} = \{0, \mathbf{H}\}$, then $\mathcal{N}$ is called a trivial nest, otherwise it is called a nontrivial nest.

The *nest algebra* associated with $\mathcal{N}$ is the set

$$\mathcal{T}(\mathcal{N}) = \{T \in \mathcal{B}(\mathbf{H}) | T(N) \subseteq N \quad \text{for all} \quad N \in \mathcal{N}\}.$$

A nontrivial nest algebra is a triangular algebra. Indeed, if $N \in \mathcal{N} \backslash \{0, \mathbf{H}\}$ and E is the orthogonal projection onto N, then $\mathcal{N}_1 = E(\mathcal{N})$ and $\mathcal{N}_2 = (1-E)(\mathcal{N})$ are nests of N and $N^{\perp}$, respectively. Moreover, $\mathcal{T}(\mathcal{N}_1) = E\mathcal{T}(\mathcal{N})E$, $\mathcal{T}(\mathcal{N}_2) = (1-E)\mathcal{T}(\mathcal{N})(1-E)$ are nest algebras and

$$\mathcal{T}(\mathcal{N}) = \begin{bmatrix} \mathcal{T}(\mathcal{N}_1) & E\mathcal{T}(\mathcal{N})(1-E) \\ O & \mathcal{T}(\mathcal{N}_2) \end{bmatrix}.$$

Note that any finite dimensional nest algebra is isomorphic to a complex block upper triangular matrix algebra. We refer the reader to [10] for the theory of nest algebras.

3. Commuting mappings of generalized matrix algebras

Throughout this section, we denote the generalized matrix algebra of order 2 originated from the Morita context $(A, B, M, N, \Phi_{MN}, \Psi_{NM})$ by

$$\mathcal{G} := \begin{bmatrix} A & M \\ N & B \end{bmatrix},$$

where at least one of the two bimodules M and N is distinct from zero. We always assume that M is faithful as a left A-module and also as a right B-module, but no any constraint conditions on N. In view of the aforementioned isomorphism relation between generalized matrix algebras of order 2 and generalized matrix algebras of order n ($n > 2$) (see below Section 2.2) and technical considerations, only generalized matrix algebras of order 2 are considered in this paper. $Z(\mathcal{G})$ represents the center of the generalized matrix algebra $\mathcal{G}$. We denote the commutator or the lie product of the elements $X, Y \in \mathcal{G}$ by $[X, Y] = XY - YX$. An $\mathcal{R}$-linear mapping $\Theta : \mathcal{G} \longrightarrow \mathcal{G}$ is said to be *commuting* if $[\Theta(X), X] = 0$ for all $X \in \mathcal{G}$. Furthermore, a commuting mapping Θ of $\mathcal{G}$ is called *proper* if it is of the form

$$\Theta(X) = XC + \Omega(X), \quad \forall X \in \mathcal{G},$$

where $C \in Z(\mathcal{G})$ and Ω is an $\mathcal{R}$-linear mapping from $\mathcal{G}$ into $Z(\mathcal{G})$. A commuting mapping which is not proper is called *improper*. The subsequent two lemmas (Lemmas 3.1 and 3.2) are in essential the Proposition 3 of [8].

Lemma 3.1. *The center of $\mathcal{G}$ is*

$$Z(\mathcal{G}) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| am = mb, na = bn, \ \forall\, m \in M, \forall n \in N \right\}.$$

Proof. It follows from [14, Lemma 1] that the center $Z(\mathcal{G})$ consists of all diagonal matrices $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, where $a \in Z(A)$, $b \in Z(B)$ and $am = mb$, $na = bn$ for all $m \in M$, $n \in N$. However, in our situation which M is faithful as a left A-module and also as a right B-module, the conditions that $a \in Z(A)$ and $b \in Z(B)$ become redundant and can be deleted. Indeed, if $am = mb$ for all $m \in M$, then for any $a' \in A$ we get

$$(aa' - a'a)m = a(a'm) - a'(am) = (a'm)b - a'(mb) = 0.$$

The assumption that M is faithful as a left $\mathcal{A}$-module leads to $aa' - a'a = 0$ and hence $a \in Z(A)$. Likewise, we also have $b \in Z(B)$. $\square$

Let us define two natural $\mathcal{R}$-linear projections $\pi_A : \mathcal{G} \to A$ and $\pi_B : \mathcal{G} \to B$ by

$$\pi_A : \begin{bmatrix} a & m \\ n & b \end{bmatrix} \longmapsto a \quad \text{and} \quad \pi_B : \begin{bmatrix} a & m \\ n & b \end{bmatrix} \longmapsto b.$$

By Lemma 3.1 it is easy to see that $\pi_A(Z(\mathcal{G}))$ is a subalgebra of $Z(A)$ and that $\pi_B(Z(\mathcal{G}))$ is a subalgebra of $Z(B)$.

Lemma 3.2. *There exists a unique algebraic isomorphism $\varphi : \pi_A(Z(\mathcal{G})) \longrightarrow \pi_B(Z(\mathcal{G}))$ such that $am = m\varphi(a)$ and $na = \varphi(a)n$ for all $a \in \pi_A(Z(\mathcal{G}))$, $m \in M$, $n \in N$.*

Proof. For a fixed $a \in \pi_A(Z(\mathcal{G}))$, if $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \begin{bmatrix} a & 0 \\ 0 & b' \end{bmatrix} \in Z(\mathcal{G})$, we have $am = mb = mb'$ for any $m \in M$. Since M is faithful as a right B-module, $b = b'$. That means there exists a unique $b \in \pi_B(Z(\mathcal{G}))$, which is denoted by $\varphi(a)$, such that $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in Z(\mathcal{G})$. Thus $\begin{bmatrix} a & 0 \\ 0 & \varphi(a) \end{bmatrix}\begin{bmatrix} 0 & m \\ n & 0 \end{bmatrix} = \begin{bmatrix} 0 & m \\ n & 0 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & \varphi(a) \end{bmatrix}$ for all $m \in M$, $n \in N$. So $am = m\varphi(a)$, $na = \varphi(a)n$ for any $m \in M$, $n \in N$. We easily observe that the mapping φ is also surjective. It remains to show that φ is an algebraic isomorphism.

For any $a, a' \in \pi_A(Z(\mathcal{G}))$ and $r \in \mathcal{R}$, we have

$$(ra)m = r(am) = r(m\varphi(a)) = m(r\varphi(a)),$$
$$(a + a')m = m(\varphi(a) + \varphi(a')),$$

and

$$(aa')m = a(a'm) = (a'm)\varphi(a) = a'(m\varphi(a)) = m\varphi(a)\varphi(a').$$

Therefore $\varphi(ra) = r\varphi(a)$, $\varphi(a + a') = \varphi(a) + \varphi(a')$ and $\varphi(aa') = \varphi(a)\varphi(a')$, and these facts complete the proof of the lemma. $\square$

Before proving our main result, we must describe the general form of an arbitrary commuting mapping of the generalized matrix algebra $\mathcal{G}$.

Proposition 3.3. *Let Θ be a commuting mapping of $\mathcal{G}$. Then Θ is of the form*

$$\begin{aligned} &\Theta\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) \\ = &\begin{bmatrix} \delta_1(a) + \delta_2(m) + \delta_3(n) + \delta_4(b) & \delta_1(1)m - m\mu_1(1) \\ \mu_4(1)n - n\delta_4(1) & \mu_1(a) + \mu_2(m) + \mu_3(n) + \mu_4(b) \end{bmatrix}, \end{aligned} \qquad (\bigstar)$$

where

$$\begin{aligned} &\delta_1 : A \longrightarrow A, && \delta_2 : M \longrightarrow Z(A), && \delta_3 : N \longrightarrow Z(A), && \delta_4 : B \longrightarrow Z(A), \\ &\mu_1 : A \longrightarrow Z(B), && \mu_2 : M \longrightarrow Z(B), && \mu_3 : N \longrightarrow Z(B), && \mu_4 : B \longrightarrow B \end{aligned}$$

are all $\mathcal{R}$-linear mappings satisfying the following conditions:

(1) *δ_1 and μ_4 are commuting mappings of A and B, respectively*;
(2) $\delta_1(a)m - m\mu_1(a) = a(\delta_1(1)m - m\mu_1(1))$, $\mu_1(a)n - n\delta_1(a) = (n\delta_4(1) - \mu_4(1)n)a$;

(3) $\delta_4(b)m - m\mu_4(b) = (m\mu_1(1) - \delta_1(1)m)b$, $\mu_4(b)n - n\delta_4(b) = b(\mu_4(1)n - n\delta_4(1))$;
(4) $\delta_2(m)m = m\mu_2(m)$, $\delta_3(n)m = m\mu_3(n)$;
(5) $\mu_3(n)n = n\delta_3(n)$, $\mu_2(m)n = n\delta_2(m)$.

Proof. We assume that the commuting map Θ is of the form

$$\begin{aligned}&\Theta\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right)\\ &= \begin{bmatrix} \delta_1(a)+\delta_2(m)+\delta_3(n)+\delta_4(b) & \tau_1(a)+\tau_2(m)+\tau_3(n)+\tau_4(b) \\ \nu_1(a)+\nu_2(m)+\nu_3(n)+\nu_4(b) & \mu_1(a)+\mu_2(m)+\mu_3(n)+\mu_4(b) \end{bmatrix},\end{aligned} \tag{3.1}$$

for all $\begin{bmatrix} a & m \\ n & b \end{bmatrix} \in \mathcal{G}$, where $\delta_1, \delta_2, \delta_3, \delta_4$ are $\mathcal{R}$-linear mappings from A, M, N, B to A, respectively; $\tau_1, \tau_2, \tau_3, \tau_4$ are $\mathcal{R}$-linear mappings from A, M, N, B to M, respectively; $\nu_1, \nu_2, \nu_3, \nu_4$ are $\mathcal{R}$-linear mappings from A, M, N, B to N, respectively; $\mu_1, \mu_2, \mu_3, \mu_4$ are $\mathcal{R}$-linear mappings from A, M, N, B to B, respectively.

In view of $\left[\Theta\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right), \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right] = 0$, we routinely compute that $\tau_1(1) = \nu_1(1) = 0$. Linearizing $[\Theta(X), X] = 0$ leads to

$$[\Theta(X), Y] = [X, \Theta(Y)] \tag{3.2}$$

for all $X, Y \in \mathcal{G}$. For any $a \in A, b \in B$, taking $X = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ and $Y = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ into (3.2) yields

$$[\Theta(X), Y] = \begin{bmatrix} 0 & -\tau_1(a) - \tau_4(b) \\ \nu_1(a) + \nu_4(b) & 0 \end{bmatrix} \tag{3.3}$$

and

$$[X, \Theta(Y)] = \begin{bmatrix} [a, \delta_1(1)] & 0 \\ 0 & [b, \mu_1(1)] \end{bmatrix}. \tag{3.4}$$

Thus $-\tau_1(a) - \tau_4(b) = 0$ and $\nu_1(a) + \nu_4(b) = 0$ for all $a \in A, b \in B$. Since a and b are arbitrary elements of A and B, respectively, we know that $\tau_1 = \tau_4 = 0$ and that $\nu_1 = \nu_4 = 0$. Repeating the same computational process and taking $X = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $Y = \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}$ into (3.2), we get

$$[\Theta(X), Y] = \begin{bmatrix} 0 & \delta_1(1)m - m\mu_1(1) \\ 0 & 0 \end{bmatrix} \tag{3.5}$$

and

$$[X, \Theta(Y)] = \begin{bmatrix} 0 & \tau_2(m) \\ -\nu_2(m) & 0 \end{bmatrix}. \tag{3.6}$$

Combining (3.5) with (3.6) gives $\nu_2 = 0$ and $\tau_2(m) = \delta_1(1)m - m\mu_1(1)$ for all $m \in M$. Similarly, if we choose $X = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $Y = \begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}$, then we arrive at $\tau_3 = 0$ and $\nu_3(n) = \mu_4(1)n - n\delta_4(1)$. for all $n \in N$. Therefore (3.1) becomes

$$\begin{aligned}&\Theta\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right)\\ &= \begin{bmatrix} \delta_1(a)+\delta_2(m)+\delta_3(n)+\delta_4(b) & \delta_1(1)m - m\mu_1(1) \\ \mu_4(1)n - n\delta_4(1) & \mu_1(a)+\mu_2(m)+\mu_3(n)+\mu_4(b) \end{bmatrix}\end{aligned} \tag{3.7}$$

for all $\begin{bmatrix} a & m \\ n & b \end{bmatrix} \in \mathcal{G}$.

By (3.7) we immediately compute that

$$\begin{aligned}\begin{bmatrix}0 & 0\\ 0 & [\mu_1(a),b]\end{bmatrix} &= \left[\Theta\left(\begin{bmatrix}a & 0\\ 0 & 0\end{bmatrix}\right),\begin{bmatrix}0 & 0\\ 0 & b\end{bmatrix}\right]\\ &= \left[\begin{bmatrix}a & 0\\ 0 & 0\end{bmatrix},\Theta\left(\begin{bmatrix}0 & 0\\ 0 & b\end{bmatrix}\right)\right]\\ &= \begin{bmatrix}[a,\delta_4(b)] & 0\\ 0 & 0\end{bmatrix}\end{aligned} \tag{3.8}$$

for all $a \in A, b \in B$. So $[\mu_1(a), b] = 0$ and $[a, \delta_4(b)] = 0$ for all $a \in A, b \in B$. This gives the fact that $\mu_1(a) \in Z(B)$ and $\delta_4(b) \in Z(A)$ for all $a \in A, b \in B$. By the previous fact and (3.7) we obtain

$$\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} = \left[\Theta\left(\begin{bmatrix}a & 0\\ 0 & b\end{bmatrix}\right),\begin{bmatrix}a & 0\\ 0 & b\end{bmatrix}\right] = \begin{bmatrix}[\delta_1(a),a] & 0\\ 0 & [\mu_4(b),b]\end{bmatrix}. \tag{3.9}$$

Then $[\delta_1(a), a] = 0$ for all $a \in A$ and $[\mu_4(b), b] = 0$ for all $b \in B$. This implies that δ_1 and μ_4 are commuting mappings of A and B, respectively.

Putting $X = \begin{bmatrix}a & 0\\ 0 & 0\end{bmatrix}$ and $Y = \begin{bmatrix}0 & m\\ n & 0\end{bmatrix}$ and applying (3.7) again yields

$$[\Theta(X), Y] = \begin{bmatrix}0 & \delta_1(a)m - m\mu_1(a)\\ \mu_1(a)n - n\delta_1(a) & 0\end{bmatrix} \tag{3.10}$$

and

$$[X, \Theta(Y)] = \begin{bmatrix}[a,\delta_2(m)] + [a,\delta_3(n)] & a(\delta_1(1)m - m\mu_1(1))\\ (n\delta_4(1) - \mu_4(1)n)a & 0\end{bmatrix} \tag{3.11}$$

for all $a \in A, m \in M, n \in N$. By (3.10) and (3.11) it follows that $\delta_1(a)m - m\mu_1(a) = a(\delta_1(1)m - m\mu_1(1))$ for all $a \in A, m \in M$ and that $\mu_1(a)n - n\delta_1(a) = (n\delta_4(1) - \mu_4(1)n)a$ for all $a \in A, n \in N$, which is the required statement (2). On the other hand, we also have $[a, \delta_2(m)] + [a, \delta_3(n)] = 0$ for all $a \in A, m \in M, n \in N$. Due to the arbitrariness of the elements m and n, we get that $\delta_2(m) \in Z(A)$ for all $m \in M$ and that $\delta_3(n) \in Z(A)$ for all $n \in N$. Similarly, choosing $X = \begin{bmatrix}0 & 0\\ 0 & b\end{bmatrix}$ and $Y = \begin{bmatrix}0 & m\\ n & 0\end{bmatrix}$ we compute that

$$[\Theta(X), Y] = \begin{bmatrix}0 & \delta_4(b)m - m\mu_4(b)\\ \mu_4(b)n - n\delta_4(b) & 0\end{bmatrix} \tag{3.12}$$

and

$$[X, \Theta(Y)] = \begin{bmatrix}0 & (m\mu_1(1) - \delta_1(1)m)b\\ b(\mu_4(1)n - n\delta_4(1)) & [b,\mu_2(m)] + [b,\mu_3(n)]\end{bmatrix} \tag{3.13}$$

for all $b \in B, m \in M, n \in N$. The statement (3) of this proposition follows from (3.12) and (3.13). In view of the arbitrariness of the elements m and n, we obtain $\mu_2(m) \in Z(B)$ for all $m \in M$ and $\mu_3(n) \in Z(B)$ for all $n \in N$.

Finally, taking $X = \begin{bmatrix}0 & m\\ n & 0\end{bmatrix}$ into the relation $[\Theta(X), X] = 0$ leads to

$$\begin{aligned}0 &= [\Theta(X), X]\\ &= \begin{bmatrix}(\delta_1(1)m - m\mu_1(1))n & (\delta_2(m) + \delta_3(n))m\\ (\mu_2(m) + \mu_3(n))n & (\mu_4(1)n - n\delta_4(1))m\end{bmatrix}\\ &\quad - \begin{bmatrix}m(\mu_4(1)n - n\delta_4(1)) & m(\mu_2(m) + \mu_3(n))\\ n(\delta_2(m) + \delta_3(n)) & n(\delta_1(1)m - m\mu_1(1))\end{bmatrix}\end{aligned} \tag{3.14}$$

for all $m \in M, n \in N$. Thus $(\delta_2(m) + \delta_3(n))m = m(\mu_2(m) + \mu_3(n))$ for all $m \in M, n \in N$. Let us choose $n = 0$. Then $\delta_2(m)m = m\mu_2(m)$ for all $m \in M$ and hence $\delta_3(n)m = m\mu_3(n)$ for all $m \in M, n \in N$. On the other hand, $(\mu_2(m) + \mu_3(n))n = n(\delta_2(m) + \delta_3(n))$ for all $m \in M, n \in N$. Let us take $m = 0$. We have $\mu_3(n)n = n\delta_3(n)$ for all $n \in N$ and further get $\mu_2(m)n = n\delta_2(m)$ for all $m \in M, n \in N$. Hereto, the statements (4) and (5) are obtained. We complete the proof of the proposition. □

Lemma 3.4. *With notations as above, then $\mu_1^{-1}(\pi_B(Z(\mathcal{G})))$ and $\delta_4^{-1}(\pi_A(Z(\mathcal{G})))$ are ideals of A and B, respectively. Furthermore, $[A, A] \subseteq \mu_1^{-1}(\pi_B(Z(\mathcal{G})))$ and $[B, B] \subseteq \delta_4^{-1}(\pi_A(Z(\mathcal{G})))$.*

Proof. For any $a, a' \in A$ and any $m \in M, n \in N$, it follows from the condition (2) of Proposition 3.3 that

$$\begin{aligned} a'a(\delta_1(1)m - m\mu_1(1)) &= \delta_1(a'a)m - m\mu_1(a'a) \\ &= a'(\delta_1(a)m - m\mu_1(a)) \end{aligned}$$

and

$$\begin{aligned} (n\delta_4(1) - \mu_4(1)n)a'a &= \mu_1(a'a)n - n\delta_1(a'a) \\ &= (\mu_1(a')n - n\delta_1(a'))a. \end{aligned}$$

The previous two equalities mean that

$$\delta_1(a'a)m - m\mu_1(a'a) - a'(\delta_1(a)m - m\mu_1(a)) = 0 \tag{3.15}$$

and

$$\mu_1(a'a)n - n\delta_1(a'a) - (\mu_1(a')n - n\delta_1(a'))a = 0 \tag{3.16}$$

for all $a, a' \in A, m \in M, n \in N$. On the other hand, we know that δ_1 is a commuting mapping of A, and hence $\delta_1(1) \in Z(A)$. Proposition 3.3 has also shown that $\delta_4(1) \in Z(A)$. Thus we have

$$\begin{aligned} aa'(\delta_1(1)m - m\mu_1(1)) &= \delta_1(aa')m - m\mu_1(aa') \\ &= a(\delta_1(1)a'm - a'm\mu_1(1)) \\ &= \delta_1(a)a'm - a'm\mu_1(a) \end{aligned}$$

and

$$\begin{aligned} (n\delta_4(1) - \mu_4(1)n)aa' &= \mu_1(aa')n - n\delta_1(aa') \\ &= (na\delta_4(1) - \mu_4(1)na)a' \\ &= \mu_1(a')na - na\delta_1(a'). \end{aligned}$$

The above two equalities mean that

$$\delta_1(aa')m - m\mu_1(aa') - \delta_1(a)a'm + a'm\mu_1(a) = 0 \tag{3.17}$$

and

$$\mu_1(aa')n - n\delta_1(aa') - \mu_1(a')na + na\delta_1(a') = 0 \tag{3.18}$$

for all $a, a' \in A, m \in M, n \in N$.

Combining (3.15) with (3.17) leads to

$$\left(\delta_1([a', a]) + [\delta_1(a), a']\right) m = m\mu_1([a', a]). \tag{3.19}$$

By Proposition 3.3 we know that δ_1 is a commuting mapping of A and hence $[a, \delta_1(a')] = [\delta_1(a), a']$ for all $a, a' \in A$. Combining this fact with (3.16) and (3.18) yields

$$\mu_1([a', a])n = n\left(\delta_1([a', a]) + [\delta_1(a), a']\right). \tag{3.20}$$

It follows from the equalities (3.19), (3.20) and Lemma 3.1 that $\mu_1([a', a]) \in \pi_B(Z(\mathcal{G}))$ and hence $[A, A] \subseteq \mu_1^{-1}(\pi_B(Z(\mathcal{G})))$.

Suppose that $a \in \mu_1^{-1}(\pi_{\mathcal{B}}(Z(\mathcal{G})))$, i.e., $\mu_1(a) \in \pi_{\mathcal{B}}(Z(\mathcal{G}))$. Then by (3.15) and Lemma 3.2 we have

$$\left(\delta_1(a'a) - a'\delta_1(a) + a'\varphi^{-1}(\mu_1(a))\right) m = m\mu_1(a'a) \tag{3.21}$$

for all $a' \in A$. We by (3.18) (exchange a with a') and Lemma 3.2 arrive at

$$\mu_1(a'a)n = n\left(\delta_1(a'a) - a'\delta_1(a) + a'\varphi^{-1}(\mu_1(a))\right) \tag{3.22}$$

for all $a' \in A$. Therefore, the equalities (3.21), (3.22) and Lemma 3.1 imply that $\mu_1(a'a) \in \pi_B(Z(\mathcal{G}))$ for all $a' \in A$. Hence $\mu_1^{-1}(\pi_B(Z(\mathcal{G})))$ is a left ideal of A. On the other hand, by (3.17) and Lemma 3.2 we obtain that

$$\left(\delta_1(aa') - \delta_1(a)a' + a'\varphi^{-1}(\mu_1(a))\right) m = m\mu_1(aa') \tag{3.23}$$

for all $a' \in A$. Taking into account (3.16) (exchange a with a') and Lemma 3.2 we get

$$\mu_1(aa')n = n\left(\delta_1(aa') - \delta_1(a)a' + a'\varphi^{-1}(\mu_1(a))\right) \tag{3.24}$$

for all $a' \in A$. So, the equalities (3.23), (3.24) and Lemma 3.1 imply that $\mu_1(aa') \in \pi_B(Z(\mathcal{G}))$. This shows that $\mu_1^{-1}(\pi_B(Z(\mathcal{G})))$ is also a right ideal of A. Thus $\mu_1^{-1}(\pi_B(Z(\mathcal{G})))$ is an ideal of A

Using the same procedure we can prove that $\delta_4^{-1}(\pi_A(Z(\mathcal{G})))$ is also an ideal of B and $[B,B] \subseteq \delta_4^{-1}(\pi_A(Z(\mathcal{G})))$. $\square$

Now we consider the sufficient and necessary conditions for any commuting mapping of the generalized matrix algebra $\mathcal{G}$ to be proper.

Proposition 3.5. *Let Θ be a commuting mapping of $\mathcal{G} = \begin{bmatrix} A & M \\ N & B \end{bmatrix}$. For any $\begin{bmatrix} a & m \\ n & b \end{bmatrix} \in \mathcal{G}$, we write*

$$\begin{aligned} & \Theta\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) \\ = & \begin{bmatrix} \delta_1(a)+\delta_2(m)+\delta_3(n)+\delta_4(b) & \delta_1(1)m - m\mu_1(1) \\ \mu_4(1)n - n\delta_4(1) & \mu_1(a)+\mu_2(m)+\mu_3(n)+\mu_4(b) \end{bmatrix}. \end{aligned} \tag{$\bigstar$}$$

Then the following statements are equivalent:

(1) *Θ is proper*; *i.e. $\Theta(X) = XC + \Omega(X)$ for all $X \in \mathcal{G}$, where $C \in Z(\mathcal{G})$ and Ω maps $\mathcal{G}$ into $Z(\mathcal{G})$.*

(2) *$\mu_1(A) \subseteq \pi_B(Z(\mathcal{G}))$, $\delta_4(B) \subseteq \pi_A(Z(\mathcal{G}))$ and $\begin{bmatrix} \delta_2(m) & 0 \\ 0 & \mu_2(m) \end{bmatrix} \in Z(\mathcal{G})$, $\begin{bmatrix} \delta_3(n) & 0 \\ 0 & \mu_3(n) \end{bmatrix} \in Z(\mathcal{G})$ for all $m \in M, n \in N$.*

(3) $\delta_1(1) \in \pi_A(Z(\mathcal{G}))$, $\mu_1(1) \in \pi_B(Z(\mathcal{G}))$ and $\begin{bmatrix} \delta_2(m) & 0 \\ 0 & \mu_2(m) \end{bmatrix} \in Z(\mathcal{G})$, $\begin{bmatrix} \delta_3(n) & 0 \\ 0 & \mu_3(n) \end{bmatrix} \in Z(\mathcal{G})$ for all $m \in M, n \in N$.

Proof. (1) $\Longrightarrow$ (3). Suppose that the commuting mapping Θ on $\mathcal{G}$ is proper. Then there exists $C = \begin{bmatrix} a_1 & 0 \\ 0 & \varphi(a_1) \end{bmatrix} \in Z(\mathcal{G})$ such that $\Theta(X) = XC + \Omega(X)$ for all $X \in \mathcal{G}$, where a_1 is some element in $\pi_A(Z(\mathcal{G}))$. Let us choose $X = \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} \in \mathcal{G}$. Assume that $\Omega(X)$ is of the form $\begin{bmatrix} a_2 & 0 \\ 0 & \varphi(a_2) \end{bmatrix}$, where a_2 is some element in $\pi_A(Z(\mathcal{G}))$. By the property ($\bigstar$) of Proposition 3.3 we have

$$\Theta(X) = \begin{bmatrix} \delta_2(m) & \delta_1(1)m - m\mu_1(1) \\ 0 & \mu_2(m) \end{bmatrix} \tag{3.25}$$

for all $m \in M$. On the other hand,

$$\Theta(X) = XC + \Omega(X) = \begin{bmatrix} a_2 & m\varphi(a_1) \\ 0 & \varphi(a_2) \end{bmatrix} \tag{3.26}$$

for some two elements $a_1, a_2 \in \pi_A(Z(\mathcal{G}))$ and for all $m \in M$. Combining (3.25) with (3.26) and applying Lemma 3.2 yields

$$\begin{bmatrix} \delta_2(m) & 0 \\ 0 & \mu_2(m) \end{bmatrix} \in Z(\mathcal{G}) \tag{3.27}$$

and

$$(\delta_1(1) - a_1)m = m\mu_1(1) \tag{3.28}$$

for some element $a_1 \in \pi_A(Z(\mathcal{G}))$ and for all $m \in M$. Similarly, if we take $X = \begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix} \in \mathcal{G}$ and repeat the same computational procedure, then

$$\begin{bmatrix} \delta_3(n) & 0 \\ 0 & \mu_3(n) \end{bmatrix} \in Z(\mathcal{G}) \tag{3.29}$$

and

$$na_1 = \mu_4(1)n - n\delta_4(1) \tag{3.30}$$

will follows, where a_1 is some element in $\pi_A(Z(\mathcal{G}))$ and n is an arbitrary element in N. By the condition (2) of Proposition 3.3 the equality (3.30) is equal to

$$na_1 = \mu_4(1)n - n\delta_4(1) = n\delta_1(1) - \mu_1(1)n.$$

That is

$$n(\delta_1(1) - a_1) = \mu_1(1)n \tag{3.31}$$

for some element $a_1 \in \pi_A(Z(\mathcal{G}))$ and for all $n \in N$. Combining (3.28), (3.31) with Lemma 3.1 we get $\delta_1(1) - a_1 \in \pi_A(Z(\mathcal{G}))$ and $\mu_1(1) \in \pi_B(Z(\mathcal{G}))$. Thus $\delta_1(1) \in \pi_A(Z(\mathcal{G}))$ and $\mu_1(1) \in \pi_B(Z(\mathcal{G}))$, which is the desired result.

(3) $\Longrightarrow$ (1). Let Θ be a commuting mapping of $\mathcal{G}$ satisfying the property ($\bigstar$). Assume that

$$\Omega(X) := \Theta(X) - XC$$

for all $X \in \mathcal{G}$, where $C = \begin{bmatrix} \delta_1(1) - \varphi^{-1}(\mu_1(1)) & 0 \\ 0 & \varphi(\delta_1(1)) - \mu_1(1) \end{bmatrix} \in Z(\mathcal{G})$. We assert that $\Omega(\mathcal{G}) \subseteq Z(\mathcal{G})$. By Lemma 3.2 and the property ($\bigstar$) we obtain

$$\begin{aligned} \Omega\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = & \begin{bmatrix} \delta_1(a) - a\delta_1(1) + a\varphi^{-1}(\mu_1(1)) & 0 \\ 0 & \mu_1(a) \end{bmatrix} \\ & + \begin{bmatrix} \delta_4(b) & 0 \\ 0 & \mu_4(b) - b\varphi(\delta_1(1)) + b\mu_1(1) \end{bmatrix} \\ & + \begin{bmatrix} \delta_2(m) + \delta_3(n) & 0 \\ 0 & \mu_2(m) + \mu_3(n) \end{bmatrix} \end{aligned} \tag{3.32}$$

for all $\begin{bmatrix} a & m \\ n & b \end{bmatrix} \in \mathcal{G}$. By Lemma 3.2 and the condition (2) of Proposition 3.3 we have

$$\begin{aligned} & (\delta_1(a) - a\delta_1(1) + a\varphi^{-1}(\mu_1(1)))m - m\mu_1(a) \\ & = a(\delta_1(1)m - m\mu_1(1)) - a(\delta_1(1) - \varphi^{-1}(\mu_1(1)))m \\ & = 0 \end{aligned}$$

for all $a \in A, m \in M$. That is,

$$(\delta_1(a) - a\delta_1(1) + a\varphi^{-1}(\mu_1(1)))m = m\mu_1(a) \tag{3.33}$$

for all $a \in A, m \in M$. By Lemma 3.2 and the condition (2) of Proposition 3.3 and the fact $\delta_1(1) \in Z(A)$ we get

$$\begin{aligned} & n(\delta_1(a) - a\delta_1(1) + a\varphi^{-1}(\mu_1(1))) - \mu_1(a)n \\ & = (\mu_4(1)n - n\delta_4(1))a - na(\delta_1(1) - \varphi^{-1}(\mu_1(1))) \\ & = (n\delta_1(1) - \mu_1(1)n)a - na(\delta_1(1) - \varphi^{-1}(\mu_1(1))) \\ & = 0 \end{aligned}$$

for all $a \in A, n \in N$. That is,

$$n(\delta_1(a) - a\delta_1(1) + a\varphi^{-1}(\mu_1(1))) = \mu_1(a)n \tag{3.34}$$

for all $a \in A, n \in N$. In view of (3.33), (3.34) and Lemma 3.1 we arrive at

$$\begin{bmatrix} \delta_1(a) - a\delta_1(1) + a\varphi^{-1}(\mu_1(1)) & 0 \\ 0 & \mu_1(a) \end{bmatrix} \in Z(\mathcal{G}) \tag{3.35}$$

for all $a \in A$. Likewise, by the condition (3) of Proposition 3.3 and the fact $\mu_1(1) \in Z(B)$ we have

$$\begin{aligned} &\delta_4(b)m - m(\mu_4(b) - b\varphi(\delta_1(1)) + b\mu_1(1)) \\ &= (m\mu_1(1) - \delta_1(1)m)b + mb(\varphi(\delta_1(1)) - \mu_1(1)) \\ &= 0 \end{aligned}$$

for all $b \in B, m \in M$. This shows

$$\delta_4(b)m = m(\mu_4(b) - b\varphi(\delta_1(1)) + b\mu_1(1)) \tag{3.36}$$

for all $b \in B, m \in M$. By the conditions (2) and (3) of Proposition 3.3 we have

$$\begin{aligned} &(\mu_4(b) - b\varphi(\delta_1(1)) + b\mu_1(1))n - n\delta_4(b) \\ &= b(\mu_4(1)n - n\delta_4(1)) - b(\varphi(\delta_1(1)) - \mu_1(1))n \\ &= b(n\delta_1(1) - \mu_1(1)n) - b(\varphi(\delta_1(1)) - \mu_1(1))n \\ &= 0 \end{aligned}$$

for all $b \in B, n \in N$. This implies

$$n\delta_4(b) = (\mu_4(b) - b\varphi(\delta_1(1)) + b\mu_1(1))n \tag{3.37}$$

for all $b \in B, n \in N$. Taking into account (3.36), (3.37) and Lemma 3.1 yields

$$\begin{bmatrix} \delta_4(b) & 0 \\ 0 & \mu_4(b) - b\varphi(\delta_1(1)) + b\mu_1(1) \end{bmatrix} \in Z(\mathcal{G}) \tag{3.38}$$

for all $b \in B$. The equalities (3.35), (3.38) and the assumption $\left[\begin{smallmatrix} \delta_2(m) & 0 \\ 0 & \mu_2(m) \end{smallmatrix}\right] \in Z(\mathcal{G})$, $\left[\begin{smallmatrix} \delta_3(n) & 0 \\ 0 & \mu_3(n) \end{smallmatrix}\right] \in Z(\mathcal{G})$ lead to

$$\Omega\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) \in Z(\mathcal{G})$$

for all $\begin{bmatrix} a & m \\ n & b \end{bmatrix} \in \mathcal{G}$, which is the desired assertion.

(2) $\Longrightarrow$ (3). Since $\mu_1(1) \in \mu_1(A) \subseteq \pi_B(Z(\mathcal{G}))$, $\mu_1(1) \in \pi_B(Z(\mathcal{G}))$ is obvious. By the condition (3) of Proposition 3.3 we have $\delta_4(1)m - m\mu_4(1) = m\mu_1(1) - \delta_1(1)m$ for all $m \in M$. This leads to

$$\delta_1(1)m = m\left(\mu_1(1) + \mu_4(1) - \varphi(\delta_4(1))\right) \tag{3.39}$$

for all $m \in M$. On the other hand, in view of the condition (2) of Proposition 3.3 we obtain $\mu_1(1)n - n\delta_1(1) = n\delta_4(1) - \mu_4(1)n$ for all $n \in N$. That is

$$n\delta_1(1) = \left(\mu_1(1) + \mu_4(1) - \varphi(\delta_4(1))\right)n \tag{3.40}$$

for all $n \in N$. Thus the equalities (3.39), (3.40) and Lemma 3.1 imply that $\delta_1(1) \in \pi_A(Z(\mathcal{G}))$.

(3) $\Longrightarrow$ (2). Since $\mu_1(1) \in \pi_B(Z(\mathcal{G}))$, the ideal $\mu_1^{-1}(\pi_B(Z(\mathcal{G})))$ of A contains the identity element 1. Hence $A = \mu_1^{-1}(\pi_B(Z(\mathcal{G})))$ and $\mu_1(A) \subseteq \pi_B(Z(\mathcal{G}))$. Due to the condition (3) of Proposition 3.3 we get $\delta_4(b)m - m\mu_4(b) = (m\mu_1(1) - \delta_1(1)m)b$ for all $b \in B, m \in M$. This gives

$$\delta_4(b)m = m\left(\mu_4(b) + \mu_1(1)b - \varphi(\delta_1(1))b\right) \tag{3.41}$$

for all $b \in B, m \in M$. On the other hand, by the conditions (3) and (2) of Proposition 3.3 it follows that $\mu_4(b)n - n\delta_4(b) = b(\mu_4(1)n - n\delta_4(1)) = b(n\delta_1(1) - \mu_1(1)n)$ for all $b \in B, n \in N$. Note that the fact $\mu_1(1) \in Z(B)$. So

$$n\delta_4(b) = (\mu_4(b) + \mu_1(1)b - \varphi(\delta_1(1))b)\, n \tag{3.42}$$

for all $b \in B, n \in N$. Combining the equalities (3.41), (3.42) with Lemma 3.1 yields that $\delta_4(b) \in \pi_A(Z(\mathcal{G}))$ for all $b \in B$ and hence $\delta_4(B) \subseteq \pi_A(Z(\mathcal{G}))$. $\square$

Now we are in a position to state our main theorem. This theorem will provide a sufficient condition which enable any commuting mapping on the generalized matrix algebra $\mathcal{G}$ is proper.

Theorem 3.6. *Let Θ be a commuting mapping of $\mathcal{G}$. If the following three conditions are satisfied*:

(1) $Z(B) = \pi_B(Z(\mathcal{G}))$, *or* $A = [A, A]$;
(2) $Z(A) = \pi_A(Z(\mathcal{G}))$, *or* $B = [B, B]$;
(3) *there exist* $m_0 \in M, n_0 \in N$ *such that*

$$Z(\mathcal{G}) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| a \in Z(A), b \in Z(B), am_0 = m_0 b, n_0 a = bn_0 \right\}.$$

Then Θ is proper.

Proof. If $Z(B) = \pi_B(Z(\mathcal{G}))$, then $\mu_1(A) \subseteq \pi_B(Z(\mathcal{G}))$ by Proposition 3.3. If $A = [A, A]$, then $\mu_1^{-1}(\pi_B(Z(\mathcal{G}))) = A$ by Lemma 3.4. This implies that $\mu_1(A) \subseteq \pi_B(Z(\mathcal{G}))$. Similarly, the condition (2) will give the fact that $\delta_4(B) \subseteq \pi_A(Z(\mathcal{G}))$.

According to the condition (2) of Proposition 3.5, it is sufficient to prove that condition (3) implies that $\begin{bmatrix} \delta_2(m) & 0 \\ 0 & \mu_2(m) \end{bmatrix} \in Z(\mathcal{G})$ and $\begin{bmatrix} \delta_3(n) & 0 \\ 0 & \mu_3(n) \end{bmatrix} \in Z(\mathcal{G})$ for all $m \in M, n \in N$. By Proposition 3.3 we know that $\delta_2(m) \in Z(A)$, $\mu_2(m) \in Z(B)$ and $\delta_2(m)m = m\mu_2(m)$, $n\delta_2(m) = \mu_2(m)n$ for all $m \in M$, $n \in N$. Since $\delta_2(m_0)m_0 = m_0\mu_2(m_0)$ and $n_0\delta_2(m_0) = \mu_2(m_0)n_0$, we have $\begin{bmatrix} \delta_2(m_0) & 0 \\ 0 & \mu_2(m_0) \end{bmatrix} \in Z(\mathcal{G})$ by hypothesis. Thus $\begin{bmatrix} \delta_2(m_0) & 0 \\ 0 & \mu_2(m_0) \end{bmatrix} \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_2(m_0) & 0 \\ 0 & \mu_2(m_0) \end{bmatrix}$ for all $m \in M$. So $\delta_2(m_0)m = m\mu_2(m_0)$ for all $m \in M$. Taking into account the above relations we compute that

$$\begin{aligned} \delta_2(m_0 + m)(m_0 + m) &= \delta_2(m_0)m_0 + \delta_2(m_0)m + \delta_2(m)m_0 + \delta_2(m)m \\ &= m_0\mu_2(m_0) + m\mu_2(m_0) + \delta_2(m)m_0 + m\mu_2(m) \end{aligned} \tag{3.43}$$

and

$$\begin{aligned} \delta_2(m_0 + m)(m_0 + m) &= (m_0 + m)\mu_2(m_0 + m) \\ &= m_0\mu_2(m_0) + m\mu_2(m_0) + m_0\mu_2(m) + m\mu_2(m) \end{aligned} \tag{3.44}$$

for all $m \in M$. Combining equalities (3.43) with (3.44) leads to $\delta_2(m)m_0 = m_0\mu_2(m)$ for all $m \in M$. Note that we always have $n_0\delta_2(m) = \mu_2(m)n_0$ for all $m \in M$. And hence, $\begin{bmatrix} \delta_2(m) & 0 \\ 0 & \mu_2(m) \end{bmatrix} \in Z(\mathcal{G})$ by hypothesis.

On the other hand, by Proposition 3.3 again we know that $\delta_3(n) \in Z(A)$, $\mu_3(n) \in Z(B)$ and $\delta_3(n)m = m\mu_3(n)$, $n\delta_3(n) = \mu_3(n)n$ for all $m \in M, n \in M$. Since $\delta_3(n_0)m_0 = m_0\mu_3(n_0)$ and $n_0\delta_3(n_0) = \mu_3(n_0)n_0$, we have $\begin{bmatrix} \delta_3(n_0) & 0 \\ 0 & \mu_3(n_0) \end{bmatrix} \in Z(\mathcal{G})$ by hypothesis. Thus $\begin{bmatrix} \delta_3(n_0) & 0 \\ 0 & \mu_3(n_0) \end{bmatrix} \begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix} \begin{bmatrix} \delta_3(n_0) & 0 \\ 0 & \mu_3(n_0) \end{bmatrix}$ for all $n \in N$. So $\mu_3(n_0)n = n\delta_3(n_0)$ for all $n \in N$. Therefore we compute that

$$\begin{aligned} \mu_3(n_0 + n)(n_0 + n) &= \mu_3(n_0)n_0 + \mu_3(n_0)n + \mu_3(n)n_0 + \mu_3(n)n \\ &= n_0\delta_3(n_0) + n\delta_3(n_0) + \mu_3(n)n_0 + n\delta_3(n) \end{aligned} \tag{3.45}$$

and

$$\begin{aligned} \mu_3(n_0 + n)(n_0 + n) &= (n_0 + n)\delta_3(n_0 + n) \\ &= n_0\delta_3(n_0) + n\delta_3(n_0) + n_0\delta_3(n) + n\delta_3(n) \end{aligned} \tag{3.46}$$

for all $n \in N$. Combining (3.45) with (3.46) gives $\mu_3(n)n_0 = n_0\delta_3(n)$ for all $n \in N$. It should be noted that $\delta_3(n)m_0 = m_0\mu_3(n)$ always holds for all $n \in N$. So $\begin{bmatrix} \delta_3(n) & 0 \\ 0 & \mu_3(n) \end{bmatrix} \in Z(\mathcal{G})$ by hypothesis. We complete the proof of the theorem. $\square$

4. Applications

In this section, we will present some applications of commuting mappings to full matrix algebras, inflated algebras and triangular algebras. In addition, we introduce the notion of (α, β)-commuting mappings and of (α, β)-biderivations. And then some characterizations on (α, β)-commuting mappings and (α, β)-biderivations of one special class of nest algebras are given. We refer the reader to Section 2 for precise definitions and related terminology.

4.1. Full matrix algebras

Let $\mathcal{R}$ be a commutative ring with identity and A be a unital algebra over $\mathcal{R}$. $M_n(A)$ (resp. $M_n(\mathcal{R})$) is the full matrix algebra over A (resp. $\mathcal{R}$), where $n \geqslant 2$.

Corollary 4.1. *Any commuting mapping on the algebra $M_n(A)$ is proper.*

One can directly check that $M_n(A)$ satisfies all conditions (1)–(3) of Theorem 3.6. Therefore, every commuting mapping on $M_n(A)$ is proper. The anonymous referee pointed out to us that this corollary can also be obtained by applying the notion of FI-degree of functional identities and related results in [5].

Corollary 4.2. *Let $\mathcal{G} = \begin{bmatrix} A & M \\ N & B \end{bmatrix}$ be a generalized matrix algebra and Θ be a commuting mapping of $\mathcal{G}$. If $Z(A) = \mathcal{R}1$ and $Z(B) = \mathcal{R}1$, then Θ is of the form $\Theta(X) = rX + \Omega(X)1$, where $r \in \mathcal{R}$ and Ω is a linear functional on $\mathcal{G}$.*

Proof. It is easy to verify that $Z(\mathcal{G}) = \mathcal{R}1$, and hence $Z(A) = \pi_A(Z(\mathcal{G})), Z(B) = \pi_B(Z(\mathcal{G}))$. Note that the condition (3) of Theorem 3.6 holds for all nonzero $m \in M$ and for all nonzero $n \in N$. So $\Theta(X) = rX + \Omega(X)1$ for all $X \in \mathcal{G}$, where $r \in \mathcal{R}$ and Ω is a linear functional on $\mathcal{G}$. $\square$

Corollary 4.3. *Let $B(A, V, \gamma)$ be the inflated algebra defined in Section 2.4. If $B(A, V, \gamma)$ has an identity element, then each commuting mapping of $B(A, V, \gamma)$ is proper.*

Proof. If $B(A, V, \gamma)$ has an identity element, then the matrix Γ defined by the bilinear form γ is invertible in the full matrix algebra $M_n(A)$ by [13, Proposition 4.2]. We define

$$\begin{aligned} \sigma : M_n(A) &\longrightarrow (M_n(A), \Gamma) \\ X &\longmapsto X\Gamma^{-1}. \end{aligned}$$

Note that $\sigma(X) \circ \sigma(Y) = \sigma(X)\Gamma\sigma(Y) = XY\Gamma^{-1} = \sigma(XY)$ for all $X, Y \in M_n(A)$ and hence σ is an algebraic isomorphism. Now the result follows from Corollary 4.1 and the fact $B(A, V, \gamma) \cong (M_n(A), \Gamma)$. $\square$

4.2. Triangular algebras

Throughout this paper, we have no any constraint conditions on (B, A)-bimodule N of the generalized matrix algebra $\mathcal{G} = \begin{bmatrix} A & M \\ N & B \end{bmatrix}$. That means that N may be a zero module, and then the generalized matrix algebra $\mathcal{G}$ degenerates to a standard *triangular algebra*. The commuting mappings of triangular algebras have been considered by Cheung in [8]. Correspondingly, all results of [8] can be contained in our present work.

Corollary 4.4 ([8, Corollary 6]). *Any commuting mapping on the upper (resp. lower) triangular matrix algebra $\mathcal{T}_n(\mathcal{R})$ (resp. $\mathcal{T}_n'(\mathcal{R})$) is of the form $X \mapsto rX + \Omega(X)1$, where $r \in \mathcal{R}$ and Ω is a linear functional on $T_n(\mathcal{R})$.*

Corollary 4.5 ([8, Corollary 7]). *Any commuting mapping on nest algebras is of the form $X \mapsto rX + \Omega(X)1$, where $r \in \mathbb{C}$ and Ω is a linear functional.*

Mathieu and Runde [16] proved that every commuting derivation on a Banach algebra A maps itself into its Jacobson radical $\mathrm{Rad}(A)$. Of course, this result is valid for commuting derivations of triangular Banach algebras. We shall interpret this fact with the help of the general description of any commuting mapping and that of any derivation. The argument is somewhat different and more direct.

Theorem 4.6. *Let $\mathcal{T} = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$ be a triangular Banach algebra with respect to the norm defined by*

$$\left\| \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \right\|_{\mathcal{T}} = \|a\|_A + \|m\|_M + \|b\|_B, \quad \forall \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \in \mathcal{T}.$$

Any commuting derivation of $\mathcal{T}$ maps itself into its Jacobson radical $\mathrm{Rad}(\mathcal{T})$.

Let Θ be a commuting derivation on $\mathcal{T}$. How does Θ maps $\mathcal{T}$ into its Jacobson radical $\mathrm{Rad}(\mathcal{T})$? By the property of (★) and [7, Theorem 2.2.1] we know that

$$\Theta\left(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}\right) = \begin{bmatrix} \delta_1(a) & 0 \\ 0 & \mu_4(b) \end{bmatrix}$$

for all $\begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \in \mathcal{T}$, where δ_1 is a commuting derivation of the Banach algebra A and μ_4 is a commuting derivation of the Banach algebra B. By Mathieu and Runde's theorem [16] it follows that $\delta_1(A) \subseteq \mathrm{Rad}(A)$ and $\mu_4(B) \subseteq \mathrm{Rad}(B)$. Haghany and Varadarajan [11] showed that the Jacobson radical $\mathrm{Rad}(\mathcal{T})$ of triangular algebra $\mathcal{T}$ is

$$\begin{bmatrix} \mathrm{Rad}(A) & M \\ 0 & \mathrm{Rad}(B) \end{bmatrix}.$$

Therefore

$$\Theta\left(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}\right) \in \mathrm{Rad}(\mathcal{T})$$

for all $\begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \in \mathcal{T}$.

4.3. (α, β)-Commuting mappings of nest algebras

We conclude this section by introducing a common generalization of commuting mappings which is the so-called (α, β)-commuting mapping and giving its elementary applications to nest algebras.

Let $\mathcal{R}$ be a commutative ring with identity, A be a unital associative algebra over a commutative ring $\mathcal{R}$ and $Z(A)$ be its center. Let us denote the commutator or the lie product of the elements $a, b \in A$ by $[x, y] = ab - ba$. Let α, β be two $\mathcal{R}$-linear automorphisms of A. Recall that an $\mathcal{R}$-linear mapping $f : A \longrightarrow A$ is said to be (α, β)-*commuting* if $f(a)\alpha(a) = \beta(a)f(a)$ for all $a \in A$. In particular, when α, β are both identical mappings of A, then f becomes a commuting mapping of A, that is, $[f(a), a] = 0$ for all $a \in A$.

Let α, β be two $\mathcal{R}$-linear automorphisms of A. An $\mathcal{R}$-linear mapping $d : A \longrightarrow A$ is called an (α, β)-*derivation* if $d(ab) = d(a)\alpha(b) + \beta(a)d(b)$ for all $a, b \in A$. A bilinear $\mathcal{R}$-mapping $\Delta : A \times A \longrightarrow A$ is an (α, β)-*biderivation* if it is an (α, β)-derivation with respect to each component, meaning that

$$\Delta(ab, c) = \Delta(a, c)\alpha(b) + \beta(a)\Delta(b, c) \quad \text{and} \quad \Delta(a, bc) = \Delta(a, b)\alpha(c) + \beta(b)\Delta(a, c)$$

for all $a, b, c \in A$. In particular, when α, β are both identical mappings of A, then Δ is said to be a *biderivation*. If A is a noncommutative algebra, then the bilinear $\mathcal{R}$-mapping

$$\begin{aligned} \Delta : A \times A &\longrightarrow A \\ (a, b) &\longmapsto z[a, b], \quad \forall a, b \in A \end{aligned} \tag{$\spadesuit$}$$

is a basic example of biderivation, where $z \in Z(A)$. Biderivations of the form ($\spadesuit$) are called *inner biderivations*. Benkovič [3] investigated biderivations of triangular algebras and also provided some sufficient conditions what enable a biderivation to be inner.

The notion of (α, β)-commuting mappings is closely connected with the notion of (α, β)-biderivations. If f is an arbitrary (α, β)-commuting mapping of A, then the bilinear $\mathcal{R}$-mapping

$$\begin{aligned} \Delta : A \times A &\longrightarrow A \\ (a, b) &\longmapsto f(a)\alpha(b) - \beta(b)f(a), \quad \forall a, b \in A \end{aligned}$$

is an (α, β)-biderivation of A. Indeed, since f is (α, β)-commuting on $A, f(a)\alpha(a) = \beta(a)f(a)$ for all $a, b \in A$. Linearization of the identity $f(a)\alpha(a) = \beta(a)f(a)$ leads to

$$\Delta(a, b) = f(a)\alpha(b) - \beta(b)f(a) = \beta(a)f(b) - f(b)\alpha(a)$$

for all $a, b \in A$. Thus we have

$$\begin{aligned} \Delta(ab, c) &= f(ab)\alpha(c) - \beta(c)f(ab) \\ &= \beta(ab)f(c) - f(c)\alpha(ab) \end{aligned} \tag{4.1}$$

for all $a, b, c \in A$. On the other hand

$$\begin{aligned} \Delta(a, c)\alpha(b) + \beta(a)\Delta(b, c) &= [\beta(a)f(c) - f(c)\alpha(a)]\alpha(b) + \beta(a)[\beta(b)f(c) - f(c)\alpha(b)] \\ &= \beta(ab)f(c) - f(c)\alpha(ab) \end{aligned} \tag{4.2}$$

for all $a, b, c \in A$. In view of (4.1) and (4.2) we obtain

$$\Delta(ab, c) = \Delta(a, c)\alpha(b) + \beta(a)\Delta(b, c) \tag{4.3}$$

for all $a, b, c \in A$. Similarly, we also get

$$\Delta(a, bc) = \Delta(a, b)\alpha(c) + \beta(b)\Delta(a, c) \tag{4.4}$$

for all $a, b, c \in A$. The relations (4.3) and (4.4) imply that Δ is an (α, β)-biderivation of A.

Let $\mathbf{H}$ be a complex separable Hilbert space and $\mathcal{B}(\mathbf{H})$ be the algebra of all bounded linear operators on $\mathbf{H}$. Suppose that $\mathcal{N}$ is a nest of closed subspaces of $\mathbf{H}$ and $\mathcal{T}(\mathcal{N})$ the nest algebra associated with $\mathcal{N}$. For each element $N \in \mathcal{N}$, let us set

$$N_+ = \cap\{M \in \mathcal{N} : M \supseteq N\} \quad \text{and} \quad N_- = \cup\{M \in \mathcal{N} : M \subseteq N\},$$

where $\cap$ denotes the intersection and $\cup$ denotes the closed linear span. It follows from [10] that the commutator of $\mathcal{T}(\mathcal{N})$ is $\mathbb{C}I$. The following proposition is a generalized version of [20, Theorem 2.1].

Proposition 4.7. *Let $\mathcal{N}$ be a nest on a complex separable Hilbert space $\mathbf{H}$ and $\dim 0_+ \neq 1$ or $\dim H_-^{\perp} \neq 1$. Suppose that α, β are automorphisms of $\mathcal{T}(\mathcal{N})$ and that Δ is a nonzero (α, β)-biderivation of $\mathcal{T}(\mathcal{N})$. Then there exists an invertible linear operator $S \in \mathcal{B}(\mathbf{H})$ such that $\beta(T) = \alpha(S)\alpha(T)\alpha(S)^{-1}$ and $\Delta(T, T') = \alpha(S)\alpha([T, T'])$ for all $T, T' \in \mathcal{T}(\mathcal{N})$.*

Proof. Let Δ be a nonzero (α, β)-biderivation of $\mathcal{T}(\mathcal{N})$. Then

$$\Delta(UV, W) = \Delta(U, W)\alpha(V) + \beta(U)\Delta(V, W) \tag{4.5}$$

for all $U, V, W \in \mathcal{T}(\mathcal{N})$. Similarly, we also get

$$\Delta(U, VW) = \Delta(U, V)\alpha(W) + \beta(V)\Delta(U, W) \tag{4.6}$$

for all $U, V, W \in \mathcal{T}(\mathcal{N})$. Applying the automorphism α^{-1} to two-sides of (4.5) and (4.6) yields

$$(\alpha^{-1}\Delta)(UV, W) = (\alpha^{-1}\Delta)(U, W)V + (\alpha^{-1}\beta)(U)(\alpha^{-1}\Delta)(V, W)$$

and

$$(\alpha^{-1}\Delta)(U, VW) = (\alpha^{-1}\Delta)(U, V)W + (\alpha^{-1}\beta)(V)(\alpha^{-1}\Delta)(U, W)$$

for all $U, V, W \in \mathcal{T}(\mathcal{N})$, respectively. This shows that $\alpha^{-1}\Delta$ is an $(1, \alpha^{-1}\beta)$-biderivation of A. By [20, Theorem 2.1] we know that there exists an invertible linear operator $S \in \mathcal{B}(\mathbf{H})$ such that $(\alpha^{-1}\beta)(T) = STS^{-1}$ and $(\alpha^{-1}\Delta)[T, T'] = S[T, T']$ for all $T, T' \in \mathcal{T}(\mathcal{N})$. That is, $\beta(T) = \alpha(S)\alpha(T)\alpha(S)^{-1}$ and $\Delta(T, T') = \alpha(S)\alpha([T, T'])$ for all $T, T' \in \mathcal{T}(\mathcal{N})$, which is the desired result. $\square$

The following theorem is due to Proposition 4.7.

Theorem 4.8. *Let $\mathcal{N}$ be a nest on a complex separable Hilbert space $\mathbf{H}$ and $\dim 0_+ \neq 1$ or $\dim H_-^\perp \neq 1$. Suppose that α, β are automorphisms of $\mathcal{T}(\mathcal{N})$ and that f is an (α, β)-commuting mapping of $\mathcal{T}(\mathcal{N})$. Then there exists an invertible linear operator $S \in \mathcal{B}(\mathbf{H})$ and a $\mathbb{C}$-linear mapping $\psi : \mathcal{T}(\mathcal{N}) \longrightarrow \mathbb{C}I$ such that $\beta(T) = \alpha(S)\alpha(T)\alpha(S)^{-1}$ and $f(T) = \alpha(S)\alpha(T) + \alpha(\psi(T))\alpha(S)$ or $f(T) = \alpha(S)\alpha(\psi(T))$ for all $T \in \mathcal{T}(\mathcal{N})$*

Proof. Suppose that $\Delta \neq 0$. Then for all $T, T' \in \mathcal{T}(\mathcal{N})$, it is easy to verify that $\Delta(T, T') = f(T)\alpha(T') - \beta(T')f(T)$ is an (α, β)-biderivation of $\mathcal{T}(\mathcal{N})$. This means that $(\alpha^{-1}\Delta)(T, T') = (\alpha^{-1}f)(T)T' - (\alpha^{-1}\beta)(T')(\alpha^{-1}f)(T)$ is an $(1, \alpha^{-1}\beta)$-biderivation of $\mathcal{T}(\mathcal{N})$. Then by [20, Theorem 2.1] it follows that there exists an invertible linear operator $S \in \mathcal{B}(\mathbf{H})$ such that $(\alpha^{-1}\beta)(T) = STS^{-1}$ and $(\alpha^{-1}\Delta)[T, T'] = S[T, T']$ for all $T, T' \in \mathcal{T}(\mathcal{N})$. Therefore

$$\begin{aligned}[T, T'] &= S^{-1}(\alpha^{-1}\Delta)[T, T'] \\ &= S^{-1}((\alpha^{-1}f)(T)T' - ST'S^{-1}(\alpha^{-1}f)(T)) \\ &= S^{-1}(\alpha^{-1}f)(T)T' - T'S^{-1}(\alpha^{-1}f)(T)\end{aligned}$$

for all $T, T' \in \mathcal{T}(\mathcal{N})$. This shows that

$$(S^{-1}(\alpha^{-1}f)(T) - T)T' = T'(S^{-1}(\alpha^{-1}f)(T) - T) \tag{4.7}$$

for all $T, T' \in \mathcal{T}(\mathcal{N})$. Let us set $\psi(T) = S^{-1}(\alpha^{-1}f)(T) - T$. Then (4.7) implies that $\psi(T) \in \mathbb{C}I$ for all $T \in \mathcal{T}(\mathcal{N})$. So

$$(\alpha^{-1}f)(T) = ST + \psi(T)S$$

for all $T \in \mathcal{T}(\mathcal{N})$. That is

$$f(T) = \alpha(S)\alpha(T) + \alpha(\psi(T))\alpha(S)$$

for all $T \in \mathcal{T}(\mathcal{N})$.

If $\Delta = 0$, then $f(T)\alpha(T') = \beta(T')f(T)$ for all $T, T' \in \mathcal{T}(\mathcal{N})$. This implies that $(\alpha^{-1}f)(T)T' = (\alpha^{-1}\beta)(T')(\alpha^{-1}f)(T)$ for all $T, T' \in \mathcal{T}(\mathcal{N})$. By [10] we know that there exists an invertible linear operator $S \in \mathcal{B}(\mathbf{H})$ such that $(\alpha^{-1}\beta)(T) = STS^{-1}$ for all $T \in \mathcal{T}(\mathcal{N})$. Thus

$$S^{-1}(\alpha^{-1}f)(T)T' = T'S^{-1}(\alpha^{-1}f)(T) \tag{4.8}$$

for all $T, T' \in \mathcal{T}(\mathcal{N})$. Let us write $\psi(T) = S^{-1}(\alpha^{-1}f)(T)$. Then (4.8) means that $\psi(T) \in \mathbb{C}I$ for all $T \in \mathcal{T}(\mathcal{N})$. So

$$f(T) = \alpha(S)\alpha(\psi(T))$$

for all $T \in \mathcal{T}(\mathcal{N})$. $\square$

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